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STOCHASTIC FEYNMAN-KAC EQUATIONS ASSOCIATED TO LÉVY-ITÔ DIFFUSIONS

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ABSTRACT. We consider linear parabolic stochastic integro-PDE's of Feynman-Kac type associated to Lévy-Itô diffusions. The solution of such equations can be represented as certain Feynman-Kac functionals of the associated diffusion such that taking expectation yields the deterministic Feynamn-Kac formula. We interpret the problem in the framework of white noise analysis and consider generalized solutions in the Kondratiev distribution space. This concept allows for relaxed assumptions on the coefficients in the equations, identically to those required in problems of similar deterministic integro-PDE's.

1. INTRODUCTION

We examine solutions of linear parabolic stochastic integro-PDE's of the following type

(1.1)
$$\begin{cases} -du(t,y) = \mathcal{L}u(t,y)dt + \int_{\mathbb{R}_0} \mathcal{B}u(t,y)\nu(d\zeta)dt + g(t,y)dt \\ \qquad + \{\mathcal{L}'u(t,y) + f(t,y)\}\, dB_t + \int_{\mathbb{R}_0} \{\mathcal{B}'u(t,y) + k(t,y,\zeta)\}\, \widetilde{N}(dt,d\zeta), \\ \\ u(T,y) = \varphi(y), \quad (t,y) \in [0,T] \times \mathbb{R}, \end{cases}$$

where

$$\begin{aligned} \mathcal{L}u(t,y) &= \tfrac{1}{2}(\sigma^2(t,y) + \hat{\sigma}^2(t,y))\, \partial_{yy}u(t,y) + b(t,y)\, \partial_y u(t,y) + c(t,y)u(t,y) \\ \mathcal{B}u(t,y) &= u(t,y+\gamma(t,y,\zeta)) + u(t,y+\hat{\gamma}(t,y,\zeta)) - 2u(t,y) \\ \mathcal{L}'u(t,y) &= \sigma(t,y)\, \partial_y u(t,y) + p(t,y)u(t,y) \\ \mathcal{B}'u(t,y) &= u(t,y+\gamma(t,y,\zeta)) - u(t,y) + q(t,y,\zeta)u(t,y), \end{aligned}$$

and where B_t is Brownian motion and $\widetilde{N}(dt,d\zeta) = N(dt,d\zeta) - dt\nu(d\zeta)$ is the compensated jump measure of a pure jump Lévy process L_t. Without the jump parts $\int_{\mathbb{R}_0} \mathcal{B}u(t,y)\nu(d\zeta)$ and $\int_{\mathbb{R}_0} \{\mathcal{B}'u(t,y) + k(t,y,\zeta)\}\, \widetilde{N}(dt,d\zeta)$ these equations have been studied by many authors, see f. ex. [K3], [Pa], [KR2]. The solutions of such equations can be represented as Feynman-Kac

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functionals of certain associated Itô diffusions such that taking expectation (which make the sochastic integrals diappear in equation (1.1) yields the usual deteministic Feynman-Kac formula. However, in order to ensure the meaning of equation (1.1) in the classical solution framework one has to require smoothness assumptions on the coeeficients that are much stronger than those which guarantee the existence of the corresponding Feynamn-Kac functional solution candidate. This gap is reduced in the work of [MR2] where the authors introduce the notion of so called soft solutions (see also [MR1]). This concept extends solutions in standard approaches to stochastic partial differential equations and the main tool used here is the Cameron-Martin-Itô theory of Wiener chaos (see [CM], [K1]).

In this paper we interpret equation (1.1) in the white noise framework for Lévy processes developed in [ØP], [LP] and [LØP], which is the analogue for Lévy processes of the white noise theory for Brownian motion in [HØUZ]. Using white noise notations one can rewrite equation (1.1) as equation (3.1) in Section 3. In the white noise framework the notion of a solution is extended to the concept of generalized solutions who take values in the *Kondratiev distribution space* $(\mathcal{S})_{-1}$ (see Section 2 for definitions). Due to this concept of solution one is able to reduce the assumptions needed on the coefficients in equation (1.1) to those common for similar deterministic integro-PDE problems. It is shown that as in the continuous case the explicit solution can be represented as a Feynman-Kac type functional of a certain associated Lévy-Itô diffusion (where however the integration is w.r.t. a two parameter Lévy process).

The study of stochastic integro-PDEs of the type (1.1) is as in the continuous case motivated by its appearence in different applications. One important example is the *Zakai equation* occuring in non-linear filtering problems for jump diffusions. This equation is a special case of equation (1.1) and describes the unnormalized density of the filter. For more details regarding the *Zakai equation* for jump diffusions and its solution see [MP]. A second important example of equation (1.1) is referred to as *backward diffusion equation.* The continuous version goes back to [Za] and is further treated in e.g. [KR1], [K2] and [R]. It states that if $Y_s^{t,y}$ solves the Itô diffusion described by

$$\begin{cases} dY_s = b(Y_{s-})ds + \sigma(Y_{s-})dB_s \\ Y_t = y,\ t \leq s \leq T, \end{cases}$$

then under appropriate smoothness conditions on the coefficients $u(t,y) = Y_T^{1-t,y}$ is a solution of

$$\begin{cases} du(t,y) = \left\{\frac{1}{2}\sigma^2(y)\,\partial_{yy}u(t,y) + b(y)\,\partial_y u(t,y)\right\}dt + \sigma(y)\,\partial_y u(t,y)d\overleftarrow{B_t} \\ u(0,y) = y. \end{cases}$$

Here $\overleftarrow{B_t}$ is the Brownian motion $B_T - B_{T-t}$. In the jump case there is an analogue *backward diffusion equation for Lévy-Itô diffusions* which we derive in Section 3 as a Corollary.

In the remaining parts of the paper we recapitulate the essential concepts of white noise theory for Lévy processes in Section 2 before these are used to state and solve the stocastic Feynman-Kac problem in Section 3.

2. White noise framework

In this Section we provide a brief review of some concepts of a white noise theory for Lévy processes, developed in [ØP], [LP] and [LØP]. In the next Section we will use this theory as a basic tool to determine the solution of a SPDE in the Lévy-Hida space. For general information about white noise theory the reader is referred to the excellent accounts of [HKPS], [Ku] and [O].

Let us recall that a *Lévy process* $L(t)$ is a stochastic process on $\mathbb{R}_+$, which has independent and stationary increments starting at zero, i.e. $L(0) = 0$. The process $L(t)$ is by its nature a càdlàg semimartingale, which is uniquely determined by the characteristic triplet

$$(B_t, C_t, \hat{\mu}) = (a \cdot t, \sigma \cdot t, \nu(d\zeta)dt), \tag{2.1}$$

where a, σ are constants and where ν is the *Lévy measure* on $\mathbb{R}_0 := \mathbb{R} - \{0\}$. For more information about Lévy processes we refer to e.g. [B], [Sa] or [JS].

We first concentrate on the white noise framework in the case of a pure jump Lévy processes. i.e. $L(t)$ has no Brownian motion part. We denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space on $\mathbb{R}^d$. The space $\mathcal{S}'(\mathbb{R}^d)$ is the dual of $\mathcal{S}(\mathbb{R}^d)$, that is the space of tempered distributions. We want to work with a white noise measure, which is constructed on the nuclear algebra $\widetilde{\mathcal{S}}'(X)$, introduced in [LØP]. The space $\widetilde{\mathcal{S}}(X)$ is defined as the quotient algebra

$$\widetilde{\mathcal{S}}(X) = \mathcal{S}(X)/\mathcal{N}_\pi, \tag{2.1}$$

where $\mathcal{S}(X)$ is a subspace of $\mathcal{S}(X)$, given by

$$\mathcal{S}(X) := \left\{ \varphi(t,\zeta) \in \mathcal{S}(\mathbb{R}^2) : \varphi(t,0) = (\frac{\partial}{\partial \zeta}\varphi)(t,0) = 0 \right\} \tag{2.2}$$

and where the closed ideal $\mathcal{N}_\pi$ in $\mathcal{S}(X)$ is defined as

$$\mathcal{N}_\pi := \{\phi \in \mathcal{S}(X) : \|\phi\|_{L^2(\pi)} = 0\} \tag{2.3}$$

with $\pi = \nu(d\zeta)dt$. The space $\widetilde{\mathcal{S}}(X)$ is a (countably Hilbertian) nuclear algebra. We indicate by $\widetilde{\mathcal{S}}'(X)$ its dual.

From the Bochner-Minlos theorem we deduce that there exists a unique probability measure μ on the Borel sets of $\widetilde{\mathcal{S}}'(X)$ such that

$$\int_{\widetilde{\mathcal{S}}'(X)} e^{i\langle\omega,\phi\rangle} d\mu(\omega) = \exp\left(\int_X (e^{i\phi} - 1) d\pi\right) \tag{2.4}$$

for all $\phi \in \widetilde{\mathcal{S}}(X)$, where $\langle\omega, \phi\rangle := \omega(\phi)$ denotes the action of $\omega \in \widetilde{\mathcal{S}}'(X)$ on $\phi \in \widetilde{\mathcal{S}}(X)$. The measure μ on $\Omega = \widetilde{\mathcal{S}}'(X)$ is called *(pure jump) Lévy white noise probability measure.*

In the sequel we consider the compensated Poisson random measure induced through relation 2.4

$$\widetilde{N}(dt, d\zeta) = N(dt, d\zeta) - \nu(d\zeta)dt$$

associated with a Lévy process $L(t)$, which is defined on the *white noise probability space*

$$(\Omega, \mathcal{F}, P) = \left(\widetilde{\mathcal{S}}'(X), \mathcal{B}(\widetilde{\mathcal{S}}'(X)), \mu\right).$$

By using generalized Charlier polynomials $C_n(\omega) \in \left(\widetilde{\mathcal{S}}(X)^{\widehat{\otimes} n}\right)'$ (dual of the n-th completed symmetric tensor product of $\widetilde{\mathcal{S}}(X)$ with itself) it is possible to construct an orthogonal $L^2(\mu)-$basis $\{\mathcal{K}_\alpha(\omega)\}_{\alpha\in\mathcal{J}}$ defined by

$$\mathcal{K}_\alpha(\omega) = \left\langle C_{|\alpha|}(\omega), \delta^{\widehat{\otimes}\alpha} \right\rangle, \tag{2.5}$$

where $\mathcal{J}$ is the multiindex set of all $\alpha = (\alpha_1, \alpha_2, ...)$ with finitely many non-zero components $\alpha_i \in \mathbb{N}_0$. The symbol $\delta^{\widehat{\otimes}\alpha}$ denotes the symmetrization of $\delta_1^{\otimes\alpha_1} \otimes ... \otimes \delta_j^{\otimes\alpha_j}$, where $\{\delta_j\}_{j\geq 1} \subset \widetilde{\mathcal{S}}(X)$ is an orthonormal basis of $L^2\left(\mathbb{R} \times \mathbb{R}_0, dt\nu(d\zeta)\right)$.

So every $X \in L^2(\mu)$ has the unique representation

$$X = \sum_{\alpha\in\mathcal{J}} c_\alpha \mathcal{K}_\alpha$$

with Fourier coefficients $c_\alpha \in \mathbb{R}$. Moreover we have the isometry

$$\|X\|^2_{L^2(\mu)} = \sum_{\alpha\in\mathcal{J}} \alpha! c_\alpha^2 \tag{2.6}$$

with $\alpha! := \alpha_1!\ \alpha_1!...$ for $\alpha \in \mathcal{J}$. The *Kondratiev test function space* $(\mathcal{S})_1$ consists of all $f = \sum_{\alpha\in\mathcal{J}} c_\alpha \mathcal{K}_\alpha \in L^2(\mu)$ such that

$$\|f\|^2_{1,k} := \sum_{\gamma\in\mathcal{J}^m} (\alpha!)^2\, c_\alpha^2 (2\mathbb{N})^{k\alpha} < \infty \tag{2.7}$$

holds for all $k \in \mathbb{N}_0$ with weights $(2\mathbb{N})^{k\alpha} = (2 \cdot 1)^{k\alpha_1}(2 \cdot 2)^{k\alpha_2}...(2 \cdot l)^{k\alpha_l}$, if $Index(\alpha) := l$. The space $(\mathcal{S})_1$ is given the projective topology, induced by the norms $(\|\cdot\|_{1,k})_{k\in\mathbb{N}_0}$ in (2.8). The *Kondratiev distribution space*, denoted by $(\mathcal{S})_{-1}$ is the topological dual of $(\mathcal{S})_1$. So we obtain the following Gel'fand triple

$$(\mathcal{S})_1 \hookrightarrow L^2(\mu) \hookrightarrow (\mathcal{S})_{-1}. \tag{2.8}$$

We can endow $(\mathcal{S})_{-1}$ with the structure of a topological algebra by introducing the *Wick product* $\diamond$, defined by

$$(\mathcal{K}_\alpha \diamond \mathcal{K}_\beta)(\omega) = (\mathcal{K}_{\alpha+\beta})(\omega), \ \alpha, \beta \in \mathcal{J} \tag{2.9}$$

The product is linearly extensible to $(\mathcal{S})_{-1} \times (\mathcal{S})_{-1}$. It can be proven e.g. that

$$\langle C_n(\omega), f_n \rangle \diamond \langle C_m(\omega), g_m \rangle = \big\langle C_{n+m}(\omega), f_n \widehat{\otimes} g_m \big\rangle \tag{2.10}$$

for $f_n \in \widetilde{\mathcal{S}}(X)^{\widehat{\otimes} n}$and $g_m \in \widetilde{\mathcal{S}}(X)^{\widehat{\otimes} m}$ (see [LØP]).

A nice feature of the Lévy-Hida distribution space is that it carries the white noise $\overset{\bullet}{\widetilde{N}}(t,\zeta)$ of the Poisson random measure $\widetilde{N}(dt, d\zeta)$. That is the formal Radon-Nikodym derivative of $\widetilde{N}(dt, d\zeta)$ defined as

$$\overset{\bullet}{\widetilde{N}}(t,\zeta) = \sum_{k \geq 1} \delta_k(t,\zeta) \mathcal{K}_{\epsilon_k}(\omega) \tag{2.11}$$

is in $(\mathcal{S})_{-1}$ $dt\nu(d\zeta)-$a.e.. The Wick product relates to stochastic integrals w.r.t. to $\widetilde{N}(dt, d\zeta)$ in the following way: If $Y(t, \zeta, \omega)$ is a predictable process, fulfilling the condition $E \int_0^T \int_{\mathbb{R}_0} Y^2(t, \zeta, \omega) dt\nu(d\zeta) < \infty$, then $Y(t, \zeta, \omega) \diamond \overset{\bullet}{\widetilde{N}}(t,\zeta)$ is $\lambda \times \nu$-Bochner integrable in $(\mathcal{S})_{-1}$ and

$$\int_0^T \int_{\mathbb{R}_0} Y(t,\zeta,\omega) \ \widetilde{N}(dt, d\zeta) = \int_0^T \int_{\mathbb{R}_0} Y(t,\zeta,\omega) \diamond \overset{\bullet}{\widetilde{N}}(t,\zeta) dt\nu(d\zeta). \tag{2.12}$$

An analogous relation is also valid for the Brownian motion. See [LØP] or [ØP] for definitions.

One of our main tools in the study of Lévy-Itô diffusions is the *Lévy Hermite transform* $\mathcal{H}$, which is used to give a characterization of distributions in $(\mathcal{S})_{-1}$ (see characterization theorem 2.3.8 in [LØP]). Similar to the Gaussian case the definition of $\mathcal{H}$ rests on the basis $\{\mathcal{K}_\alpha(\omega)\}_{\alpha \in \mathcal{J}}$ in (2.6). The *Lévy Hermite transform* of $X(\omega) = \sum_\alpha c_\alpha \mathcal{K}_\alpha(\omega) \in (\mathcal{S})_{-1}$, denoted by $\mathcal{H}X$ or $\widetilde{X}$, is defined by

$$\mathcal{H}X(z) = \widetilde{X}(z) = \sum_\alpha c_\alpha z^\alpha \in \mathbb{C} \ , \tag{2.13}$$

where $z = (z_1, z_2, ...) \in \mathbb{C}^{\mathbb{N}}$, i.e. in the space of $\mathbb{C}-$valued sequences, and where $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} ....$ We have that $\mathcal{H}X(z)$ in (2.13) is absolutely convergent on the infinite dimensional neighbourhood

$$\mathbb{K}_q(R) := \left\{ (z_1, z_2, ...) \in \mathbb{C}^{\mathbb{N}} : \sum_{\alpha \neq 0} |z^{\alpha}|^2 (2\mathbb{N})^{q\alpha} < R^2 \right\} \tag{2.14}$$

for some $0 < q \leq R < \infty$. For example, the Hermite transform of $\overset{\bullet}{\widetilde{N}}(t,\zeta)$ can be evaluated as

$$\mathcal{H}(\overset{\bullet}{\widetilde{N}}(t,\zeta))(z) = \sum_{k \geq 1} \delta_k(t,\zeta) z_k. \tag{2.15}$$

The Hermite transform translates the Wick product into an ordinary (complex) product, that is

$$\mathcal{H}(X \diamond Y)(z) = \mathcal{H}(X)(z) \cdot \mathcal{H}(Y)(z). \tag{2.16}$$

As a consequence of theorem 2.3.8 in [LØP] the last relation can be generalized to Wick versions of complex analytical functions g: If the function $g : \mathbb{C} \longrightarrow \mathbb{C}$ can be expanded into a Taylor series around $\xi_0 = \mathcal{H}(X)(0)$ with real valued coefficients, then there exists a unique distribution $Y \in (\mathcal{S})_{-1}$ such that

$$\mathcal{H}(Y)(z) = g\left(\mathcal{H}(X)(z)\right) \tag{2.17}$$

on $\mathbb{K}_q(R)$ for some $0 < q \leq R < \infty$. We set $g^{\diamond}(X) = Y$.

For example, the Wick version of the exponential function exp can be written as

$$\exp^{\diamond} X = \sum_{n \geq 0} \frac{1}{n!} X^{\diamond n}. \tag{2.18}$$

Let us now shortly outline how the preceding concepts and results can be generalized to capture the case of Lévy processes with Brownian motion and pure jump part (see [P]). Indicate by μ_G the Gaussian white noise measure on the measurable space

$$(\Omega_G, \mathcal{F}_G) = (\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R}))).$$

Further recall the construction of the orthogonal $L^2(\mu_G)$ basis $\{H_\alpha(\omega)\}_{\alpha \in J}$, given by

$$H_\alpha(\omega) = \prod_{j \geq 1} h_{\alpha_j}(\langle \omega, \xi_j \rangle),$$

where $\langle \omega, \cdot \rangle = \omega(\cdot)$ and where ξ_j resp. $h_j, j = 1, 2, ...$ are the Hermite functions resp. Hermite polynomials. Using μ_J to denote the pure jump white noise measure on $(\Omega_J, \mathcal{F}_J) = (\widetilde{\mathcal{S}}'(X), \mathcal{B}(\widetilde{\mathcal{S}}'(X)))$, we can define the *Lévy white noise measure* μ as the product measure $\mu_G \times \mu_J$ on

$$(\Omega, \mathcal{F}) = (\Omega_G \times \Omega_J, \mathcal{F}_G \otimes \mathcal{F}_J). \tag{2.19}$$

Set

$$\mathcal{L}_\gamma(\omega) = \mathcal{L}_\gamma(\omega_1, \omega_2) = H_\alpha(\omega_1)\mathcal{K}_\beta(\omega_2), \tag{2.20}$$

if $\gamma = (\alpha, \beta) \in \mathcal{I} : = \mathcal{J}^2$. Thus $(\mathcal{L}_\gamma(\omega)_{\gamma\in\mathcal{I}}$ constitutes an $L^2(\mu)-$basis with norm expression

$$\|\mathcal{L}_\gamma\|^2_{L^2(\mu)} = \gamma!,$$

where $\gamma! := \alpha!\beta!$ for $\gamma = (\alpha, \beta) \in \mathcal{I}$.

As in the the pure jump setting, we employ the basis $(\mathcal{L}_\gamma(\omega)_{\gamma\in\mathcal{I}}$ to establish the concepts of Hida space, Wick product or Hermite transform to the mixture of Gaussian and pure jump Lévy noise. As in [HØUZ], the white noise $\overset{\bullet}{B}_t$ of Brownian motion is defined as an element in the Hida distribution space

$$\overset{\bullet}{B}_t := \sum_k \xi_k(t) H_{\epsilon_k}. \tag{2.19}$$

We conclude this Section with two remarks. First note that by choosing an appropriate basis the above described white noise theory can be established on any time interval $[0, T]$ instead of the complete time line $\mathbb{R}$ (which is used in the next section). Second, due to notaional convenience we have chosen to present the white noise framework only for Lévy processes with one dimensional time parameter. The generalization to d-parameter Lévy processes (d-dimensional time parameter), which are used in the beginning of the next Section, is straight forward and can be found in [P].

3. Generalized Solutions of Stochastic Feynman-Kac Equations associated to Lévy-Itô Diffusions

Let $(\Omega, \mathcal{F}, \mu)$ be a white noise space corresponding to a fixed space-time interval $[0, T]\times$ $[0, U]$ with associated Brownian motion B_t and 2-parameter pure jump Lévy process $P(t, u)$ with Lévy measure $\nu'(d\zeta)$. For convenience we suppose that $P(t, u)$ is a square integrable 2-parameter martingale, i.e. we have the represantation

$$P(t, u) = \int_0^t \int_0^u \int_{\mathbb{R}_0} \zeta\, \widetilde{M}(dt, du, d\zeta),$$

where $\widetilde{M}(dt, du, d\zeta) = M(dt, du, d\zeta) - dtdu\nu'(d\zeta)$ is the compensated jump measure of $P(t, u)$. The image measure of $M(dt, du, d\zeta)$ under the projection

$$p : [0, T] \times [0, U] \times \mathbb{R}_0 \longrightarrow [0, T] \times \mathbb{R}_0,$$

denoted by $N(dt, d\zeta)$, is the jump measure of a square integrable Lévy process L_t with Lévy measure $\nu(d\zeta) = U \cdot \nu'(d\zeta)$. We have the Lévy-Itô representation

$$L_t = \int_0^t \int_{\mathbb{R}_0} \zeta\, \widetilde{N}(dt, d\zeta),$$

where $\widetilde{N}(dt,d\zeta) = N(dt,d\zeta) - dt\nu(d\zeta)$ is the compensated Poisson random measure of L_t. Let $\{\mathcal{F}_t\}_{0\le t\le T}$ be the completion of the filtration generated by B_t and the Poisson random measure $N(dt,d\zeta)$. Restriction to $\mathcal{F}_T$-functionals in the white noise setting on $(\Omega,\mathcal{F},\mu)$ leads in a natural way to a white noise space $(\Omega,\mathcal{F}_T,\mu)$ corresponding to the time interval $[0,T]$ with associated Brownian motion B_t and pure jump Lévy process L_t. In the sequel this will be our underlying white noise probability space, and we consider the following linear parabolic stochastic integro-PDE

(3.1)
$$\begin{cases} 0 = \partial_t u(t,y) + \mathcal{L}u(t,y) + \int_{\mathbb{R}_0} \mathcal{B}u(t,y)\nu(d\zeta) + g(t,y) \\ + \{\mathcal{L}'u(t,y) + f(t,y)\} \diamond \overset{\bullet}{B}_t + \int_{\mathbb{R}_0} \{\mathcal{B}'u(t,y) + k(t,y,\zeta)\} \diamond \overset{\bullet}{\widetilde{N}}(t,\zeta)\nu(d\zeta) \\ u(T,y) = \varphi(y), \quad (t,y) \in [0,T]\times\mathbb{R}, \end{cases}$$

where

$$\begin{aligned} \mathcal{L}u(t,y) &= \frac{1}{2}(\sigma^2(t,y) + \hat{\sigma}^2(t,y))\,\partial_{yy}u(t,y) + b(t,y)\,\partial_y u(t,y) + c(t,y)u(t,y) \\ \mathcal{B}u(t,y) &= u(t,y+\gamma(t,y,\zeta)) + u(t,y+\hat{\gamma}(t,y,\zeta)) - 2u(t,y) \\ \mathcal{L}'u(t,y) &= \sigma(t,y)\,\partial_y u(t,y) + p(t,y)u(t,y) \\ \mathcal{B}'u(t,y) &= u(t,y+\gamma(t,y,\zeta)) - u(t,y) + q(t,y,\zeta)u(t,y). \end{aligned}$$

For notational convenience we focus in this paper on the one-dimensional case of equation (3.1), but note that the analog techniques and results go through in the n-dimensional case.

Concerning the coefficients in equation (3.1) we suppose first of all the following boundedness and Lipschitz conditions. We assume that there exist $K > 0$ and $\rho : \mathbb{R} \longrightarrow \mathbb{R}_+$ with $\int_{\mathbb{R}_0} \rho(\zeta)\nu(d\zeta) < +\infty$ such that for all $s \in [0,T]$ and $v, y \in \mathbb{R}$

$$\begin{aligned} &|b(s,y)| + |\sigma(s,y)| + |\hat{\sigma}(s,y)| + |c(s,y)| \\ &\qquad + |p(s,y)| + |g(s,y)| + |f(s,y)| \le K \quad (3.2) \\ &|\gamma(s,y,\zeta)| + |\hat{\gamma}(s,y,\zeta)| + |q(s,y,\zeta)| + |k(s,y,\zeta)| \le \rho(\zeta)\cdot K. \quad (3.3) \end{aligned}$$

and

$$\begin{aligned} &|b(s,v) - b(s,y)| + |\sigma(s,v) - \sigma(s,y)| + |\hat{\sigma}(s,v) - \hat{\sigma}(s,y)| \\ &\qquad + |p(s,v) - p(s,y)| + |g(s,v) - g(s,y)| \\ &\qquad + |f(s,v) - f(s,y)| + |\varphi(v) - \varphi(y)| \le K\,|v-y| \quad (3.4) \\ &|\gamma(s,v,\zeta) - \gamma(s,y,\zeta)| + |\hat{\gamma}(s,v,\zeta) - \hat{\gamma}(s,y,\zeta)| \\ &\qquad + |q(s,v,\zeta) - q(s,y,\zeta)| + |k(s,v,\zeta) - k(s,y,\zeta)| \le \rho(\zeta)\,|v-y|. \quad (3.5) \end{aligned}$$

In addition we set $0 \le q(s,y,\zeta)$ for all $(s,y,\zeta) \in [0,T]\times\mathbb{R}\times\mathbb{R}_0$.

Further we assume the coefficients in equation (3.1) such that with

$$\phi'_z(t) = \mathcal{H}(\overset{\bullet}{B}(t))(z) = \sum_k z_k \xi_k(t)$$

and

$$\phi_z''(t,\zeta) = \mathcal{H}(\overset{\bullet}{\tilde{N}}(t,\zeta))(z) = \sum_k z_k \delta_k(t,\zeta)$$

for a given $z \in \mathbb{C}_c^{\mathbb{N}}$ the deterministic intergro-PDE
(3.6)

$$\begin{cases} 0 = \partial_t u(t,y) + \mathcal{L}u(t,y) + \int_{\mathbb{R}_0} \mathcal{B}u(t,y)\nu(d\zeta) + g(t,y) \\ + \{\mathcal{L}'u(t,y) + f(t,y)\}\,\phi_z'(t) + \int_{\mathbb{R}_0} \{\mathcal{B}'u(t,y) + k(t,y,\zeta)\}\,\phi_z''(t,\zeta)\nu(d\zeta) \\ u(T,y) = \varphi(y), \quad (t,y) \in [0,T]\times\mathbb{R}, \end{cases}$$

has a solution u^* in $C^{1,2}([0,T],\mathbb{R})$. Sufficient criteria for this are f. ex. found in [GS] or [PH]. In ([PH], Section 5.2) the author gives a set of sufficient conditions which only implies Lipschitz regularities, which illustrates that sufficient criteria do not necessarily require the coefficients to be smooth.

Finally we suppose for a $\gamma \in (0,1)$

$$\begin{aligned} \left(g(t,y) + f(t,y) + \int_{\mathbb{R}_0} k(t,y,\zeta)\zeta\nu(d\zeta)\right) &\in C^{1,\gamma}([0,T]\times D) \\ \varphi(y) &\in C^{2+\gamma}(D), \end{aligned}$$

and that the following Hölder regularity is valid

$$||u^*||_{C^{1,2+\gamma}(G)} \le const.\cdot ||\mathcal{N}u^*||_{C^{1,\gamma}(G)} + ||\varphi||_{C^{2+\gamma}(\partial G)} \tag{3.7}$$

for every open and bounded $G = (0,T)\times D \subset \mathbb{R}_+ \times \mathbb{R}$. Here $C^{1,\gamma}$ (resp. $C^{2+\gamma}$) stands for functions who are continuously differentiable with respect to t and γ-Hölder continuous in y (resp. whose partial derivatives up to order 2 are γ-Hölder continuous). $\mathcal{N}$ denotes the operator given through

$$\begin{aligned} \mathcal{N}u &= \partial_t u(t,y) + \mathcal{L}u(t,y) + \int_{\mathbb{R}_0} \mathcal{B}u(t,y)\nu(d\zeta) \\ &\quad + \mathcal{L}'u(t,y)\phi_z'(t) + \int_{\mathbb{R}_0} \mathcal{B}'u(t,y)\phi_z''(t,\zeta)\nu(d\zeta). \end{aligned}$$

For criteria for the validity of the above Hölder regularity see f. ex. [MPr1] and the references therein.

Now let $\hat{B}_s$ be an auxiliary independent Brownian motion and $\hat{L}_t$ an auxiliary independent Lévy process, also with Lévy measure $\nu(d\zeta)$ and jump measure denoted by $\hat{N}(dt,d\zeta)$, and consider the following SDE

$$\begin{aligned} dY_s &= [b(s,Y_{s-}) - \sigma(s,Y_{s-})p(t,Y_{s-})]\,ds + \sigma(s,Y_{s-})dB_s + \hat{\sigma}(s,Y_{s-})d\hat{B}_s \\ &\quad + \int_{[0,U]}\int_{\mathbb{R}_0} \gamma(s,Y_{s^-},\zeta)1_{\left\{u\le \frac{U}{1+q(s,Y_{s^-},\zeta)}\right\}}\, M(ds,du,d\zeta) \\ &\quad + \int_{\mathbb{R}_0} \hat{\gamma}(s,Y_{s^-},\zeta)\ \hat{N}(ds,d\zeta), \\ Y_t &= y,\ t\le s\le T. \end{aligned} \tag{3.8}$$

By conditions (3.2)-(3.5) one sees like in [F] that there exists a (unique) càdlàg square integrable solution of (3.8) for all $0 \le t \le s \le T$, which we denote by $Y_s^{t,y} \in L^2$. Using this solution we define the following Feynman-Kac functional

$$
\begin{aligned}
u(t,y,\omega) \quad = \quad & E\Bigg[\varphi(Y_T^{t,y})\rho(t,T) + \int_t^T g(s,Y_s^{t,y})\rho(t,s)ds \\
& + \int_t^T f(s,Y_s^{t,y})\rho(t,s)dB_s \qquad (3.9) \\
& + \int_t^T \int_{\mathbb{R}_0} k(s,Y_s^{t,y},\zeta)\rho(t,s)\,\tilde{N}(ds,d\zeta) \Bigg| \mathcal{F}_T\Bigg], \qquad (3.10)
\end{aligned}
$$

where

$$
\begin{aligned}
\rho(t,s) \quad = \quad & \exp\Bigg\{\int_t^s c(r,Y_r^{t,y})dr + \int_t^s p(r,Y_r^{t,y})dB_r - \frac{1}{2}\int_t^s p^2(r,Y_r^{t,y})dr \\
& + \int_t^s \int_{\mathbb{R}_0} \log\left(1+q(r,Y_r^{t,y},\zeta)\right)\tilde{N}(ds,d\zeta) \\
& + \int_t^s \int_{\mathbb{R}_0} \left(\log\left(1+q(r,Y_r^{t,y},\zeta)\right) - q(r,Y_r^{t,y},\zeta)\right)\nu(d\zeta)dr\Bigg\}.
\end{aligned}
$$

Our main result in this paper is then:

Theorem 3.1. *Under the above formulated conditions we have that $u(t,y,\omega)$ as defined in 3.10 solves the stochastic integro-PDE 3.1 in the topology of $(\mathcal{S})_{-1}$.*

Before we give the proof of theorem 3.1 we state the following help lemma as given in [P].

Lemma 3.2. *Let G be a bounded open subset of $\mathbb{R}_+ \times \mathbb{R}$. Assume a process $U : G \to (\mathcal{S})_{-1}$ with $\mathcal{H}U = u$ such that u and its partial derivatives $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial y^2}$ are bounded on $G \times \mathbf{K}_q(R)$, continuous with respect to $(t,y) \in G$ for all $z \in \mathbf{K}_q(R)$, and analytic in $z \in \mathbf{K}_q(R)$ for all $(t,y) \in G$, $q < \infty$, $R > 0$. Then on $\mathbf{K}_q(R)$*

$$
\mathcal{H}\left(\frac{\partial U}{\partial t}\right) = \frac{\partial u}{\partial t}, \quad \mathcal{H}\left(\frac{\partial U}{\partial y}\right) = \frac{\partial u}{\partial y}, \quad \mathcal{H}\left(\frac{\partial^2 U}{\partial y^2}\right) = \frac{\partial u}{\partial y^2}.
$$

Proof. (Theorem 3.1) In this proof we only focus on the pure jump part of the problem, i.e. we set $\sigma(t,y), \hat{\sigma}(t,y))\,, b(t,y), f(t,y)$ and $p(t,y)$ identically to 0. The proof of the general case follows the same principle and doesn't add anything new to the existing literature. First note (see Theorem 2.7.10 in [HØUZ]) that since $u(t,y,\omega)$ is an $L^2(\mu)$ functional of L_t the Hermite transform can be expressed as

(3.11)

$$
\tilde{u}(t,y,z) := \mathcal{H}\left(u(t,y,\omega)\right)(z) = E\left[u(t,y,\omega)\cdot\mathcal{E}\left(\int_0^T \int_{\mathbb{R}_0} \phi_z''(t,\zeta)\widetilde{N}(dt,d\zeta)\right)\right],
$$

where $\phi''_z(t,\zeta) = \sum_k z_k \delta_k(t,\zeta)$ is as in (3.6), $z \in \mathbb{C}_c^{\mathbb{N}}$, and where the exponential martingale is given by

(3.12)

$$\begin{aligned}
&\mathcal{E}\left(\int_0^T \int_{\mathbb{R}_0} \phi''_z(t,\zeta)\widetilde{N}(dt,d\zeta)\right) \\
&\quad = \exp\left\{\int_0^T \int_{\mathbb{R}_0} \log(1+\phi''_z(t,\zeta))N(dt,d\zeta) - \int_0^T \int_{\mathbb{R}_0} \phi''_z(t,\zeta)\nu(d\zeta)dt\right\}.
\end{aligned}$$

In the following it is sufficient to consider the real part of 3.11, that is $z \in \mathbb{R}_c^{\mathbb{N}}$. Further we observe that we can rewrite (3.12) in terms of integration w.r.t. $M(dt,du,d\zeta) - dtdu\nu'(d\zeta)$

$$\begin{aligned}
&\mathcal{E}\left(\int_0^T \int_{\mathbb{R}_0} \phi''_z(t,\zeta)\widetilde{N}(dt,d\zeta)\right) \\
&\qquad = \exp\left\{\int_0^T \int_0^U \int_{\mathbb{R}_0} \log(1+\phi''_z(t,\zeta))M(dt,du,d\zeta)\right. \\
&\qquad\qquad \left. - \int_0^T \int_0^U \int_{\mathbb{R}_0} \phi''_z(t,\zeta)dtdu\nu'(d\zeta)\right\}.
\end{aligned}$$

By means of the Girsanov theorem for random measures (see f. ex. [JS]) we get that the Hermite transform $\tilde{u}(t,y,z)$ can be written as

$$\begin{aligned}
\tilde{u}(t,y,z) &= E\Bigg[\Bigg\{\varphi(Y_T^{t,y})\rho(t,T) + \int_t^T g(s,Y_s^{t,y})\rho(t,s)ds \\
&\qquad + \int_t^T \int_{\mathbb{R}_0} k(s,Y_s^{t,y},\zeta)\rho(t,s)\,\tilde{N}(ds,d\zeta)\Bigg\} \\
&\qquad\qquad \mathcal{E}\left(\int_0^T \int_{\mathbb{R}_0} \phi''_z(t,\zeta)\widetilde{N}(dt,d\zeta)\right)\Bigg] \\
&= E\Bigg[\varphi(\check{Y}_T^{t,y})\check{\rho}(t,T) + \int_t^T g(s,\check{Y}_s^{t,y})\check{\rho}(t,s)ds \\
&\qquad + \int_t^T \int_{\mathbb{R}_0} k(s,\check{Y}_s^{t,y},\zeta)\check{\rho}(t,s)\,\phi''_z(s,\zeta)ds\nu(d\zeta)\Bigg]
\end{aligned}$$

where $\check{Y}_s^{t,y}$ is as in (3.8) just that the jump measure $M(dt,du,d\zeta)$ now has the predictable compensator $(1+q(s,y,\zeta))\,(1+\phi''_z(s,\zeta))dsdu\nu'(d\zeta)$, and where

$$\check{\rho}(t,s) = \exp\left\{\int_t^s \left(c(r,\check{Y}_r^{t,y}) + \int_{\mathbb{R}_0} q(r,\check{Y}_r^{t,y},\zeta)\phi''_z(r,\zeta)\nu(d\zeta)\right)dr\right\}.$$

The infinitesimal generator of $\check{Y}_s^{t,y}$ applied to $\theta \in C_b^1(\mathbb{R})$ is given by

$$\begin{aligned}
\mathcal{K}_s\theta(y) &= b(s,y)\partial_y\theta(y) + \int_{\mathbb{R}_0} \{\theta(y+\hat{\gamma}(s,y,\zeta)) - \theta(y)\}\,\nu(d\zeta) \\
&\quad + \int_{[0,U]}\int_{\mathbb{R}_0} \{\theta(y+\gamma(s,y,\zeta)) - \theta(y)\}\, 1_{\left\{u \le \frac{U}{1+q(s,y,\zeta)}\right\}} \\
&\qquad\qquad (1+q(s,y,\zeta))\,(1+\phi_z''(s,\zeta))du\nu'(d\zeta) \\
&= b(s,y)\partial_y\theta(y) + \int_{\mathbb{R}_0} \{\theta(y+\gamma(s,y,\zeta)) - \theta(y)\}\,\phi_z''(s,\zeta)\nu(d\zeta) \\
&\quad + \int_{\mathbb{R}_0} \{\theta(y+\hat{\gamma}(s,y,\zeta)) + \theta(y+\gamma(s,y,\zeta)) - 2\theta(y)\}\,\nu(d\zeta).
\end{aligned}$$

So by our assumptions on the coefficients and the Feynman-Kac formula $\tilde{u}(t,y,z)$ is the solution of (3.6). The last step is to show that we can extract the Hermite transform in equation (3.6), that is to interchange Hermite transform and differentiation respectively Hermite transform and integration, and in this way end up with equation (3.1). To this end it is sufficient to show that on a neighbourhood $\mathbf{K}_q(R)$

$$\partial_y\tilde{u}(t,y,z) = \mathcal{H}\left(\partial_y u(t,y,\omega)\right) \tag{3.13}$$

and

(3.14)

$$\begin{aligned}
&\int_{\mathbb{R}_0} \sup_{z\in\mathbf{K}_q(R)} \{\tilde{u}(t,y+\gamma(s,y,\zeta),z) + \tilde{u}(t,y+\hat{\gamma}(s,y,\zeta),z) - 2\tilde{u}(t,y,z)\}\,\nu(d\zeta) \\
&\qquad + \int_{\mathbb{R}_0} \sup_{z\in\mathbf{K}_q(R)} \{\tilde{u}(t,y+\gamma(s,y,\zeta),z) - \tilde{u}(t,y,z)\}\,\phi_z''(s,\zeta)\nu(d\zeta) < \infty.
\end{aligned}$$

Relation (3.13) follows immediately from the Hölder regularity (3.7) and Lemma 3.2. Concerning relation (3.14) it is not difficult to see with the help of conditions (3.4)-(3.5) and estimates like f. ex. in ([PH], Lemma 3.1) that $\tilde{u}(t,y,z)$ is Lipschitz continuous in y uniformly on a neighbourhood $\mathbf{K}_q(R)$, i.e. for all $t \in [0,T]$

$$\sup_{z\in\mathbf{K}_q(R)} |\tilde{u}(t,y,z) - \tilde{u}(t,v,z)| \le const \cdot |y-v| \ .$$

This together with the property of $\phi_z''(s,\zeta)$ as the Hermite transform of $\overset{\bullet}{\tilde{N}}(t,\zeta)$ easily yields relation (3.14). ∎

As a corollary we now get the backward jump diffusion equation for Lévy-Itô diffusions whose Brownian motion version has its origin in [Za]. Let $Y_s^{t,y}$

denote the solution of

$$
\begin{aligned}
(3.15)\qquad dY_s &= b(Y_{s-})ds + \sigma(Y_{s-})dB_s + \gamma(Y_{s-})dL_s \\
&= b(Y_{s-})ds + \sigma(Y_{s-})dB_s + \int_{\mathbb{R}_0} \gamma(Y_{s-})\,\zeta\,\widetilde{N}(ds,d\zeta), \\
Y_t &= y,\ t \le s \le T.
\end{aligned}
$$

Then we get the correspondance of $Y_s^{t,y}$ to the following SIPDE:

Corollary 3.3. *If we set $u(t,y,\omega) := Y_T^{T-t,y}$ then, under the specified assumptions on the coefficients, $u(t,y,\omega)$ solves the following stochastic integro-PDE in the topology of $(\mathcal{S})_{-1}$*

(3.16)

$$
\begin{cases}
\partial_t u(t,y) = \frac{1}{2}\sigma^2(y)\,\partial_{yy}u(t,y) + b(y)\,\partial_y u(t,y) + \sigma(y)\,\partial_y u(t,y) \diamond \overset{\bullet}{B}_{T-t} \\
\qquad\qquad + \int_{\mathbb{R}_0} \{u(t,y+\gamma(y)\zeta) - u(t,y) - \partial_y u(t,y)\gamma(y)\zeta\}\,\nu(d\zeta) \\
\qquad\qquad\quad + \int_{\mathbb{R}_0} \{u(t,y+\gamma(y)\zeta) - u(t,y)\} \diamond \overset{\bullet}{\widetilde{N}}(T-t,\zeta)\nu(d\zeta), \\
u(0,y) = y, \quad (t,y) \in [0,T] \times \mathbb{R}.
\end{cases}
$$

Proof. The result is just a special case of Theorem 3.1 except for the following two modifications. First note that in (3.15) the jump integration is with respect to the compensated jump measure $\widetilde{N}(ds,d\zeta)$ in contrast to the integration with respect to the jump measure $M(ds,du,d\zeta)$ only in (3.8). As can be seen from the proof of Theorem 3.1 this doesn yield any problem as long as $q(t,y,\zeta) = 0$ and just causes the extra term $\partial_y u(t,y)\gamma(y)\zeta$ under the integral with respect to $\nu(d\zeta)$ in (3.16). Further, in order to obtain an initial condition rather than a terminal condition of the corresponding SIPDE on has to revert the time which leads to the time reverted white noises. ∎

Note that under the appropriate smoothness conditions on the coefficients one can integrate equation (3.16) from 0 to t and interpret it in the Itô sense

$$
\begin{cases}
du(t,y) = \left\{\frac{1}{2}\sigma^2(y)\,\partial_{yy}u(t,y) + b(y)\,\partial_y u(t,y)\right\}dt + \sigma(y)\,\partial_y u(t,y)d\overleftarrow{B_t} \\
\qquad\qquad + \int_{\mathbb{R}_0} \{u(t,y+\gamma(y)\zeta) - u(t,y) - \partial_y u(t,y)\}\,\nu(d\zeta)dt \\
\qquad\qquad\quad + \int_{\mathbb{R}_0} (u(t^-,y+\gamma(y)\zeta) - u(t^-,y))\,\overleftarrow{\widetilde{N}}(d\zeta,dt), \\
u(0,y) = y, \quad (t,y) \in [0,T) \times \mathbb{R}.
\end{cases}
$$

where $\overleftarrow{B}_t$ is the Brownian motion $B_T - B_{T-t}$ and $\overleftarrow{\widetilde{N}}(d\zeta,dt)$ is the compensated jump measure associated to the pure jump Lévy process $\overleftarrow{L}_t = L_{T^-} - L_{(T-t)^-}$ for $0 \le t < T$.

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